Supplementary File of "Push-Sum Based Algorithm for Constrained Convex Optimization Problem and Its Potential Application in Smart Grid"

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Proof 1: From the iterative rule (2d), we have

$$\bar{y}(r+1) = \bar{y}(r) + c(r) - \frac{1}{n}\sum_{i=1}^{n}\alpha(r)\partial g_i(s_i(r+1)) \tag{1}$$

where

$$c(r) = \frac{1}{n}\sum_{i\in\mathcal{N}_1(r)}\beta(r)(s_i(r+1)-x_i(r+1)) + \frac{1}{n}\sum_{i\in\mathcal{N}_2(r)}\frac{\beta(r)}{\|d_i(r+1)\|}(s_i(r+1)-x_i(r+1)).$$

Then, we can get

$$\begin{aligned}\|\bar{y}(r+1)-s^*\|^2 &= \left\|\bar{y}(r)+c(r)-\frac{1}{n}\sum_{i=1}^{n}\alpha(r)\partial g_i(s_i(r+1))-s^*\right\|^2\\ &= \|\bar{y}(r)-s^*\|^2 - \frac{2}{n}\sum_{i=1}^{n}\alpha(r)(\bar{y}(r)-s^*)^T\partial g_i(s_i(r+1))\\ &\quad + 2(\bar{y}(r)-s^*)^T c(r)\\ &\quad + \left\|c(r)-\frac{\alpha(r)}{n}\sum_{i=1}^{n}\partial g_i(s_i(r+1))\right\|^2.\end{aligned} \tag{2}$$

Then, we will further bound all terms in its right-hand side of (2). For the second term, we can further deduce that

$$\begin{aligned}&\sum_{i=1}^{n}\alpha(r)(\bar{y}(r)-s^*)^T\partial g_i(s_i(r+1))\\ &= \sum_{i=1}^{n}\alpha(r)(s_i(r+1)-s^*)^T\partial g_i(s_i(r+1))\\ &\quad + \sum_{i=1}^{n}\alpha(r)(\bar{y}(r)-s_i(r+1))^T\partial g_i(s_i(r+1))\\ &\geq \sum_{i=1}^{n}\alpha(r)(g_i(s_i(r+1))-g_i(s^*))\\ &\quad + \sum_{i=1}^{n}\alpha(r)(\bar{y}(r)-\bar{s}(r))^T\partial g_i(s_i(r+1))\\ &\quad + \sum_{i=1}^{n}\alpha(r)(\bar{s}(r)-s_i(r+1)))^T\partial g_i(s_i(r+1))\\ &= \sum_{i=1}^{n}\alpha(r)[(g_i(\bar{s}(r))-g_i(s^*))+(g_i(s_i(r+1))\\ &\quad - g_i(\bar{s}(r)))] + \sum_{i=1}^{n}\alpha(r)(\bar{y}(r)-\bar{s}(r))^T\partial g_i(s_i(r+1))\\ &\quad + \sum_{i=1}^{n}\alpha(r)(\bar{s}(r)-s_i(r+1)))^T\partial g_i(s_i(r+1)).\end{aligned} \tag{3}$$

Moreover, since the projection operator is non-expansive, we can deduce that

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\alpha(r)(\bar{y}(r)-s^*)^T\partial g_i(s_i(r+1))\\ &\geq \frac{1}{n}\sum_{i=1}^{n}\alpha(r)[(g_i(\bar{s}(r))-g_i(s^*))-\|s_i(r+1)-\bar{s}(r)\|\\ &\quad -D(\|\bar{y}(r)-\bar{s}(r)\|+\|\bar{s}(r)-s_i(r+1)\|)]\\ &\geq \alpha(r)(g(\bar{s}(r))-g(s^*))-D\alpha(r)\|\bar{y}(r)-\bar{s}(r)\|\\ &\quad -\frac{2D}{n}\sum_{i=1}^{n}\alpha(r)\|\bar{y}(r)-x_i(r+1)\|.\end{aligned} \tag{4}$$

Next, we will confirm the bound for the third term. Since the projection operator is non-expansive, we have $\|s_i(r+1)-\bar{s}(r)\| \leq \|x_i(r+1)-\bar{y}(r)\|$ and $(\bar{s}(r)-s^*)^T(\bar{s}(r)-\bar{y}(r)) \leq 0$. Thus, we have

$$\begin{aligned}&(\bar{y}(r)-s^*)^T(s_i(r+1)-x_i(r+1))\\ &= (\bar{y}(r)-\bar{s}(r))^T(\bar{s}(r)-\bar{y}(r)) + (\bar{s}(r)-s^*)^T(\bar{s}(r)-\bar{y}(r))\\ &\quad +(\bar{y}(r)-s^*)^T[(s_i(r+1)-\bar{s}(r))-(x_i(r+1)-\bar{y}(r))]\\ &\leq -\|\bar{y}(r)-\bar{s}(r)\|^2+2\|\bar{y}(r)-s^*\|\|x_i(r+1)-\bar{y}(r)\|.\end{aligned} \tag{5}$$

Therefore, based on (5), we can get

$$\begin{aligned}(\bar{y}(r)-s^*)^T c(r) \leq\ & \frac{1}{n}\sum_{i\in\mathcal{N}_1(r)}\beta(r)(-\|\bar{y}(r)-\bar{s}(r)\|^2\\ &+2\|\bar{y}(r)-s^*\|\|x_i(r+1)-\bar{y}(r)\|)\\ &\frac{1}{n}\sum_{i\in\mathcal{N}_2(r)}\frac{-\beta(r)}{\|d_i(r+1)\|}(\|\bar{y}(r)-\bar{s}(r)\|^2\\ &-2\|\bar{y}(r)-s^*\|\|x_i(r+1)-\bar{y}(r)\|).\end{aligned} \tag{6}$$

Furthermore, from (4) and [11, Lemma 1], it has $\lim_{r\to+\infty} x_i(r+1)-\bar{y}(r) = 0$. Additionally, the distant functions are continuous, which implies that

$$\lim_{r\to\infty}\left\|\frac{\bar{y}(r)-\bar{s}(r)}{\|x_i(r+1)-s_i(r+1)\|}\right\| = 1.$$

Therefore, when $\|x_i(r+1)-s_i(r+1)\| > 1$, there exist a nonnegative integer R_0 and two positive constants $\underline{l} \leq 1$ and $\bar{l}$ near to 1 such that

$$0 < \underline{l} \leq \left\|\frac{\bar{x}(r)-\bar{s}(r)}{\|d_i(r+1)\|}\right\| \leq \bar{l},\quad r \geq R_0. \tag{7}$$

Specially, we assume $R_0 = 0$ for simplicity, which has no effect on the whole analysis. Noting that

$$\begin{aligned}&2\beta(r)\|\bar{y}(r)-s^*\|\|x_i(r+1)-\bar{y}(r)\|\\ &\leq \phi_{i1}(r)\|\bar{y}(r)-s^*\|^2+\phi_{i1}(r)\end{aligned} \tag{8}$$

we have

$$\begin{aligned}&\frac{-\beta(r)}{\|d_i(r+1)\|}(\|\bar{y}(r)-\bar{s}(r)\|^2-2\|\bar{y}(r)-s^*\|\|x_i(r+1)-\bar{y}(r)\|)\\ &\leq -\beta(r)\underline{l}\|\bar{y}(r)-\bar{s}(r)\|+(\phi_{i1}(r)\|\bar{y}(r)-s^*\|^2+\phi_{i1}(r))\end{aligned} \tag{9}$$

when $i \in \mathcal{N}_2(r)$. Substituting (8) and (9) into (6) leads to

$$\begin{aligned}(\bar{y}(r)-s^*)^T c(r) \leq &-\frac{\beta(r)}{n}(|\mathcal{N}_1(r)|\|\bar{y}(r)-\bar{s}(r)\|^2+|\mathcal{N}_2(r)|\underline{l}\|\bar{y}(r)-\bar{s}(r)\|)\\ &+\frac{1}{n}\sum_{i=1}^{n}(\phi_{i1}(r)\|\bar{y}(r)-s^*\|^2+\phi_{i1}(r)).\end{aligned} \tag{10}$$

Also, we have

$$\Big\|\frac{1}{n}\sum_{i=1}^{n}\frac{\beta(r)}{\|d_i(r+1)\|}(s_i(r+1)-x_i(r+1))$$
$$-\frac{1}{n}\sum_{i=1}^{n}\alpha(r)\partial g_i(s_i(r+1))\Big\|^2$$
$$\leq 2\beta^2(r)\Big\|\frac{1}{n}\sum_{i=1}^{n}\frac{s_i(r+1)-x_i(r+1)}{\|d_i(r+1)\|}\Big\|$$
$$+2\alpha^2(r)\Big\|\frac{1}{n}\sum_{i=1}^{n}\partial g_i(s_i(r+1))\Big\|$$
$$\leq 2\beta^2(r)+2D^2\alpha^2(r). \quad (11)$$

Therefore, by combining (4), (10) and (11) with (3) together, we can further obtain

$$\|\bar{y}(r+1)-s^*\|^2 \leq \|\bar{y}(r)-s^*\|^2-2\alpha(r)(g(\bar{s}(r))-g(s^*))+2\beta^2(r)$$
$$+\frac{4D}{n}\sum_{i=1}^{n}\phi_{i2}(r)+2D\alpha(r)\|\bar{y}(r)-\bar{s}(r)\|+2D^2\alpha^2(r)$$
$$-\frac{2\beta(r)}{n}(|\mathcal{N}_1(r)|\|\bar{x}(r)-\bar{s}(r)\|^2+\underline{l}|\mathcal{N}_2(r)|\|\bar{y}(r)-\bar{s}(r)\|)$$
$$+\frac{2}{n}\sum_{i=1}^{n}\phi_{i1}(r)+\frac{2}{n}\sum_{i=1}^{n}\phi_{i1}(r)\|\bar{y}(r)-s^*\|^2. \quad (12)$$

Furthermore, it has $|\mathcal{N}_1(r)|+|\mathcal{N}_2(r)|=n$, $\forall r\in\mathbb{R}$, and thus it has

$$2D\alpha(r)\|\bar{y}(r)-\bar{s}(r)\|-\frac{2\beta(r)}{n}(|\mathcal{N}_1(r)|\|\bar{y}(r)-\bar{s}(r)\|^2$$
$$+\underline{l}|\mathcal{N}_2(r)|\|\bar{y}(r)-\bar{s}(r)\|)$$
$$=2\frac{|\mathcal{N}_1(r)|}{n}(D\alpha(r)\|\bar{y}(r)-\bar{s}(r)\|-\beta(r)\|\bar{y}(r)-\bar{s}(r)\|^2)$$
$$+2\frac{|\mathcal{N}_2(r)|}{n}(D\alpha(r)\|\bar{y}(r)-\bar{s}(r)\|-\underline{l}\beta(r)\|\bar{y}(r)-\bar{s}(r)\|)$$
$$\leq\frac{|\mathcal{N}_1(r)|}{n}\Big(D^2\frac{\alpha^2(r)}{\beta(r)}-\beta(r)\|\bar{y}(r)-\bar{s}(r)\|^2\Big)$$
$$-2\frac{\underline{l}|\mathcal{N}_2(r)|}{n}\gamma(r)\|\bar{y}(r)-\bar{s}(r)\| \quad (13)$$

where $0<\gamma(r)=\beta(r)-\frac{D}{\underline{l}}\alpha(r)$, $\forall r\in\mathbb{R}$. Therefore, it has

$$\|\bar{y}(r+1)-s^*\|^2\leq\Big(1+\frac{2}{n}\sum_{i=1}^{N}\phi_{i1}(r)\Big)\|\bar{y}(r)-s^*\|^2$$
$$-2\alpha(r)(g(\bar{s}(r))-g(s^*))-\Big(\frac{|\mathcal{N}_1(r)|}{n}\beta(r)\|\bar{y}(r)-\bar{s}(r)\|^2$$
$$+2\frac{\underline{l}|\mathcal{N}_2(r)|}{n}\gamma(r)\|\bar{y}(r)-\bar{s}(r)\|\Big)+\varphi(r). \quad (14)$$

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