Supplement Material of "A Privacy-Preserving Distributed Subgradient Algorithm for the Economic Dispatch Problem in Smart Grid"

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Proof of Theorem 1: According to iteration (10), it follows that, for any $\lambda \in \mathbb{R}^+$, one has that $\|\tau(k+1)-\lambda\|^2 = \|\tau(k) - \frac{\alpha}{m}\sum_{j=1}^m \partial_{K_j}(k) - \lambda\|^2$, from which we immediately derive that

$$\|\tau(k+1)-\lambda\|^2 \leq \|\tau(k)-\lambda\|^2 - \frac{\alpha}{m}\sum_{j=1}^m \partial_{K_j}^T(k)(\tau(k)-\lambda) + \frac{\alpha^2}{m^2}\sum_{j=1}^m \|\partial_{K_j}(k)\|^2.$$

To proceed, we consider the term $-\frac{\alpha}{m}\sum_{j=1}^m \partial_{K_j}^T(k)(\tau(k)-\lambda)$ with

$$-\frac{\alpha}{m}\sum_{j=1}^m \partial_{K_j}^T(k)(\tau(k)-\lambda) \leq \frac{\alpha}{m}\left(\sum_{j=1}^m \|\partial_{K_j}^T(k)\|\|\tau(k)-\lambda^j(k)\| - \sum_{j=1}^m \partial_{K_j}^T(k)(\lambda^j(k)-\lambda)\right).$$

According to the definition of subgradient, it also holds that

$$K_j(\lambda^j(k)) - K_j(\lambda) \geq -\|\partial_{K_j}^\tau(k)\|\|(\lambda^j(k)-\tau(k))\| + K_j(\tau) - K_j(\lambda)$$

thus we can further obtain

$$-\frac{\alpha}{m}\sum_{j=1}^m \partial_{K_j}^T(k)(\tau(k)-\lambda) \leq \frac{\alpha}{m}\sum_{j=1}^m (\|\partial_{K_j}^T(k)\| + \|\partial_{K_j}^\tau(k)\|)\|\lambda^j(k)-\tau(k)\| - \frac{\alpha}{m}(K_j(\tau(k)) - K_j(\lambda)).$$

Thus, (11) holds. ■

Proof of Theorem 2: 1) Using (10), we can obviously establish the following equation: $\tau(k) = \tau(0) - \frac{\alpha}{m}\sum_{s=0}^{k-1}\sum_{j=1}^m \partial_{K_j}(s)$. Subtracting it with (9), one can derive

$$\begin{aligned}
\|\tau(k)-\lambda^i(k)\| &\leq \max_{1\leq j\leq m} |\lambda^j(0)|\left(\sum_{j=1}^m \left|[\mathbf{W}(k-1:0)]_j^i - \frac{1}{m}\right|\right) + \phi \\
&\quad + \alpha\sum_{r=1}^k\sum_{j=1}^m \left|\frac{1}{m} - [\mathbf{W}(k-1:r)]_j^i\right| \|\partial_{K_j}(r-1)\| \\
&\quad + \frac{\alpha}{m}\sum_{j=1}^m \left\|\partial_{K_j}(k-1) - \partial_{K_i}(k)\right\| \\
&= \frac{2\alpha(R+Q)m(1+\kappa^{-N_0})}{1-\kappa^{-N_0}}\sum_{s=0}^{k-1}(1-\kappa^{-N_0})^{\frac{k-1}{N_0}} + 2\alpha Q + \phi \\
&\leq 2\alpha Q\left(1 + \frac{R+Q}{Q}\frac{m(1+\kappa^{-N_0})}{1-\kappa^{-N_0}}\sum_{s=0}^{k-1}(1-\kappa^{-N_0})^{\frac{k-s-1}{N_0}}\right) + \phi \\
&= 2\alpha Q R_1 + \phi
\end{aligned}$$

which completes the proof for 1).

2) By using (11), it follows that:

$$\|\tau(k+1)-\lambda^*\|^2 \leq \|\tau(k)-\lambda^*\|^2 + \frac{2\alpha}{m}\sum_{j=1}^m (\|\partial_{K_j}^T(k)\| + \|\partial_{K_j}^\tau(k)\|)\|\lambda^j(k)-\tau(k)\| - \frac{2\alpha}{m}[K(\tau(k)) - K^*] + \frac{\alpha^2}{m^2}Q^2.$$

Then, it follows that:

$$\|\tau(k+1)-\lambda^*\|^2 \leq \|\tau(k)-\lambda^*\|^2 + 4\alpha^2 QR_1 + m\phi - \frac{2\alpha}{m}[K(\tau(k)) - K^*] + \frac{\alpha^2}{m^2}Q^2.$$

From above inequality, one can obtain the following inequality:

$$K(\tau(k)) - K^* \leq \frac{-\|\tau(k+1)-\lambda^*\|^2 + \|\tau(k)-\lambda^*\|^2}{\frac{2\alpha}{m}} + \frac{4\alpha^2 QR_1 + m\alpha^2 Q^2 + m^2\phi}{2\alpha}.$$

We proceed to plus above inequality from $k = 0, 1, \ldots, k$ and multiply $1/k$, as a consequence, we have that

$$K(\hat{\tau}(k)) \leq \frac{m\|\tau(0)-\lambda^*\|^2}{2k\alpha} + \frac{4\alpha^2 QR_1 + m\alpha^2 Q^2 + m^2\phi}{2\alpha}.$$

In the sequel, we consider the estimate for $K(\hat{\lambda}^i(k))$, for which, according to the definition of subgradient, we have that

$$\begin{aligned}
K(\hat{\lambda}^i(k)) &\leq K(\hat{\tau}(k)) + \sum_{j=1}^m \hat{\partial}_{ij}(k)(\hat{\lambda}^i(k) - \hat{\tau}(k)) \\
&\leq K(\hat{\tau}(k)) + mQ\sum_{s=0}^{k-1}\frac{\|\lambda^i(s)-\tau(s)\|}{k} \\
&\leq \frac{m\|\tau(0)-\lambda^*\|^2}{2k\alpha} + \frac{\alpha^2 Q(4R_1 + mQ) + m^2\phi}{2\alpha} + t
\end{aligned}$$

with $t = mQ(2\alpha QR_1 + \phi)$. ■

Proof of Theorem 3: According to the multi-agent iteration (3) and the updating mechanism (7), we know that the real exchanging information over the network is the noised state $\bar{\lambda}^j(k)$ at every time instant.

We assume that the observing time window of eavesdroppers is $[0, n^*]$ with $n^* > \bar{\iota}^*$. The information obtained by it can be listed by the following equations:

$$\vec{\lambda}^i(k+1) = \begin{cases}
\sum_{j=1}^m [\mathbf{W}(k:s)]_j^i \lambda^j(s) + \sum_{r=s}^k\sum_{j=1}^m [W(k:r)]_j^i o_j(r) + o_i(k+1) \\
\quad - \alpha\sum_{r=s+1}^k\sum_{j=1}^m [\mathbf{W}(k:r)]_j^i \partial_{K_j}(r-1) - \alpha\partial_{K_i}(k), \ k < \bar{\iota}^* \\
\sum_{j=1}^m [\mathbf{W}(k:s)]_j^i \lambda^j(s) + \sum_{r=s}^k\sum_{j=1}^m [W(k:r)]_j^i o_j(r) + o_i(k+1) \\
\quad - \alpha\sum_{r=s+1}^{\bar{\iota}^*}\sum_{j=1}^m [\mathbf{W}(k:r)]_j^i \partial_{K_j}(r-1), \ \bar{\iota}^* \leq k \leq n^*.
\end{cases}$$

By Assumption 3, we assume that for agent i its neighbor k_i is honest. Thus, in above polynomial equation, the unknown information is $\{\lambda^{k_i}(s),\ s = 1, 2, \ldots, n^*\}$, $\{o_{k_i}(s),\ s = 1, 2, \ldots, n^*\}$, and $\{\partial_{K_{k_i}}(s),\ s = 1, 2, \ldots, \bar{\iota}^*\}$. There are $2n^* + \bar{\iota}^*$ unknown variables, thus we cannot infer any parameter from above equation, according to the solving approach of polynomials. ■

Conclusion: In this letter, a subgradient algorithm is proposed based on multi-agent consensus, by which an economic dispatch problem in the smart grids is considered for application to realize the minimal generating cost. The effect of hyper-parameters especially the step size on the convergence of the objective function are explored, deepening our understanding of the convergence properties of the gradient method.