Supplement Material of “A Privacy-Preserving Distributed Subgradient Algorithm for the Economic Dispatch Problem in Smart Grid”

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Proof of Theorem 1: According to iteration (10), it follows that, for any $\lambda \in \mathbb{R}^+$, one has that $||\tau(k+1) - \lambda||^2 = ||\tau(k) - \frac{m}{\alpha} \sum_{j=1}^m \partial K_j(s) - \lambda||^2$, from which we immediately derive that

$$||\tau(k+1) - \lambda||^2 \leq ||\tau(k) - \lambda||^2 - \frac{2\alpha}{m} \sum_{j=1}^m \partial K_j(s)||\tau(k) - \lambda|| + \frac{2\alpha}{m^2} \sum_{j=1}^m ||\partial K_j(s)||^2.$$ 

To proceed, we consider the term $-\frac{2\alpha}{m} \sum_{j=1}^m \partial K_j(s)(\tau(k) - \lambda)$ with

$$-\frac{2\alpha}{m} \sum_{j=1}^m \partial K_j(s)(\tau(k) - \lambda) \leq \frac{2\alpha}{m} \sum_{j=1}^m ||\partial K_j(s)|| ||\tau(k) - \lambda||.$$ 

According to the definition of subgradient, it also holds that

$$K_j(\lambda^i(k)) - K_j(\lambda) \geq -||\partial K_j(s)|| ||\lambda^i(k) - \tau(k)|| + K_j(\tau) - K_j(\lambda).$$ 

Thus, (11) holds.

Proof of Theorem 2: 1) Using (10), we can obviously establish the following equation: $\tau(k) = \tau(0) - \frac{2\alpha}{m} \sum_{s=0}^k \sum_{j=1}^m \partial K_j(s)$. Subtracting it with (9), one can derive

$$||\tau(k) - \lambda^i(k)|| \leq \max_{1 \leq j \leq m} ||\lambda^i(0)|| \sum_{j=1}^m ||W(k-1 : 0)||_j + \frac{2\alpha}{m} \sum_{s=0}^k \sum_{j=1}^m ||\partial K_j(s)|| ||\lambda^i(k) - \tau(k)|| + \frac{2\alpha}{m} \sum_{j=1}^m ||\partial K_j(s)||^2.$$ 

which completes the proof for 1).

2) By using (11), it follows that

$$||\tau(k+1) - \lambda^i||^2 \leq ||\tau(k) - \lambda^i||^2 + \frac{2\alpha}{m} \sum_{j=1}^m ||\partial K_j(s)|| ||\tau(k) - \lambda^i|| + \frac{2\alpha}{m^2} ||\partial K_j(s)||^2.$$ 

Then, it follows that:

$$||\tau(k+1) - \lambda^i||^2 \leq ||\tau(k) - \lambda^i||^2 + 4\alpha^2 QR_1 + \frac{2\alpha}{m} ||\partial K_j(s)||^2.$$

From above inequality, one can obtain the following inequality:

$$K(\tau(k)) - K^* \leq \frac{2\alpha}{m} ||\partial K_j(s)||^2 + 4\alpha^2 QR_1 + \frac{2\alpha}{m} ||\partial K_j(s)||^2.$$ 

We proceed to plus above inequality from $k = 0, 1, \ldots, k$ and multiply $1/k$, as a consequence, we have that

$$K(\hat{\tau}(k)) \leq K(\hat{\tau}(k)) + \sum_{j=1}^m \frac{1}{k} ||\partial K_j(s)||^2 + \frac{2\alpha}{m} ||\partial K_j(s)||^2 + \frac{2\alpha}{m} ||\partial K_j(s)||^2.$$ 

In the sequel, we consider the estimate for $K(\hat{\tau}(k))$, for which, according to the definition of subgradient, we have that

$$K(\hat{\tau}(k)) \leq K(\hat{\tau}(k)) + \sum_{j=1}^m \frac{1}{k} ||\partial K_j(s)||^2 + \frac{2\alpha}{m} ||\partial K_j(s)||^2 + \frac{2\alpha}{m} ||\partial K_j(s)||^2 + t.$$ 

with $t = m(2\alpha QR_1 + \phi)$.

Proof of Theorem 3: According to the multi-agent iteration (3) and the updating mechanism (7), we know that the real exchanging information over the network is the noised state $\hat{\lambda}(k)$ at every time instant.

We assume that the observing time window of eavesdroppers is $[0,n^*]$ with $n^* > t^*$. The information obtained by it can be listed by the following equations:

$$\hat{\lambda}(k+1) = \frac{\sum_{j=1}^m \sum_{s=0}^k [W(k : s)]^j \lambda^i(s) + \sum_{j=1}^m \sum_{r=s}^k [W(k : r)]^j \partial K_j(r) + \partial K_j(s+1)}{\sum_{j=1}^m \sum_{r=s}^k [W(k : r)]^j \partial K_j(r) - \alpha \partial K_j(s), s > k - t^*}.$$ 

By Assumption 3, we assume that for agent $i$ its neighbor $k_i$ is honest. Thus, in above polynomial equation, the unknown information is $\{\lambda^i(s), s = 1, 2, \ldots, n^*\}$, $\{\partial K_j(s), s = 1, 2, \ldots, n^*\}$, and $\{\partial K_j(k), k = 1, 2, \ldots, t^*\}$. There are $2n^* + t^*$ unknown variables, thus we cannot infer any parameter from above equation, according to the solving approach of polynomials.

Conclusion: In this letter, a subgradient algorithm is proposed based on multi-agent consensus, by which an economic dispatch problem in the smart grids is considered for application to realize the minimal generating cost. The effect of hyper-parameters especially the step size on the convergence of the objective function are explored, deepening our understanding of the convergence properties of the gradient method.